Response of Periodically Varying Systems to Shot Noise—Application to Switched RC Circuits

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This paper is concerned with the statistical properties of the output $y(t)$ of a periodically varying linear system when the input is random shot noise.

Usually $y(t)$ can be divided into a noise part, $y_N(t)$, and a periodic part, $y_{per}(t)$. Expressions are obtained for the Fourier components of $y_{per}(t)$ and the power spectrum of $y_N(t)$. Various averages associated with $y(t)$ are studied. Some of the results for shot noise input can be converted into corresponding results for white noise input.

Some of the theoretical results are illustrated by applying them to two examples. In both examples the system consists of an arrangement of a resistance, a condenser, and a switch which opens and closes periodically. The output is the voltage across the condenser.

I. INTRODUCTION

Consider a circuit, shown in Fig. 1a, consisting of a resistance R shunted by a switch and condenser C. The circuit is driven by a Poisson shot noise current. The elementary charges q arrive at random at an average rate of ν per second. The switch operates in a cycle with period T. It is closed during the intervals $nT < t < nT + \alpha T$ and open during the intervals $nT + \alpha T < t < (n + 1)T$ where n is an integer and $0 < \alpha \leq 1$. We are interested in the statistical properties of the voltage $V(t)$ across the condenser. In particular, we want an expression for the two-sided power spectrum $W_V(f)$ of $V(t)$.

This problem was encountered by D. D. Sell[1] during the development of a new type of spectrophotometer. The determination of an exact expression for $W_V(f)$ turned out to be unexpectedly difficult, and led to the present investigation of the more general case in which

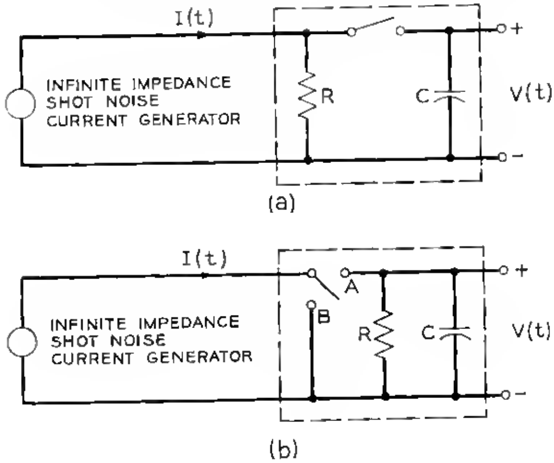

Fig. 1—RC circuits with periodically operating switch.

the switched RC circuit is replaced by a general linear network which varies periodically with time.

The systems shown in Fig. 1 are "cyclo-stationary" (this term was introduced by W. R. Bennett). Cyclo-stationary systems have been studied by a number of writers. A detailed treatment and many references are given by H. L. Hurd[2] in his thesis on periodically correlated stochastic processes. However, I have been unable to find any references dealing specifically with periodically varying systems having shot noise input. The nearest approach is contained in seven pages of anonymous handwritten notes[3] obtained by Sell. These notes give approximate results for the case of Fig. 1a with white-noise input instead of shot noise input.

In Section II, we make some general remarks about the notation and type of analysis used in this paper. Section III contains a statement of results for the general system shown in Fig. 2. In Sections IV and V, the general results are applied to the RC circuits shown in Figs. 1a and 1b. Representative curves giving $W_V(f)$ for various values of the circuit parameters in Fig. 1a are plotted in Fig. 3. Sections VI, VII, and VIII contain the derivation of the expressions stated in Section II for the various ensemble averages and the output power spectrum. The results for shot noise input can be carried over into corresponding results for white gaussian noise input. This correspondence is developed in Section IX. Appendix A gives an outline of the

analysis required in applying the general theory to get the power spectrum $W_V(f)$ of the output $V(t)$ in the RC circuit of Fig. 1a.

Roughly speaking, the shot effect formulas for a periodically varying system differ from the shot-effect formulas for a time invariant system[4] by containing an additional integration. This extra integral represents an average taken over the period.

II. REMARKS CONCERNING NOTATION AND ANALYSIS

In this paper ensemble averages are denoted by the angle bracket $\langle\ \rangle$ and time averages by over-bars. For example, consider $V(t)$ in Fig. 1. We can write $V(t) \equiv V(t, \varphi)$ where φ represents the family of random arrival times of the charges q comprising the shot noise current. When t is held fixed, $V(t)$ can be regarded as a random variable and $\langle V^l(t)\rangle$ as the average value of the lth power of $V(t)$ at time t. On the other hand, for a fixed set φ of arrival times, i.e., for a particular member of the ensemble, the time average of $V^l(t)$ is denoted by

$$\overline{V^l}(t) = \operatorname*{limit}_{T_1\to\infty} \frac{1}{T_1} \int_0^{T_1} V^l(t)\, dt. \tag{1}$$

Let $z(t)$ be an output function (e.g., $V^l(t)$) of our periodic system such that its ensemble average $\langle z(t)\rangle$ is periodic with period T, the period of the system. We assume that the time average $\overline{z(t)}$ has the same value for almost all members of the ensemble. From this assumption and the periodicity of $\langle z(t)\rangle$ it follows, upon averaging both sides of the equation

$$\overline{z(t)} = \operatorname*{limit}_{T_1\to\infty} \frac{1}{T_1} \int_0^{T_1} z(t)\, dt$$

over the ensemble, that

$$\overline{z(t)} = \frac{1}{T} \int_0^T \langle z(t)\rangle\, dt. \tag{2}$$

In addition to ensemble and time averages, we shall use E to denote expected values of time invariant random variables associated with the amplitudes of tbe shot noise impulses.

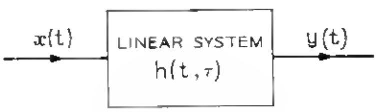

Fig. 2—Time-varying linear system specified by $h(t, \tau)$.

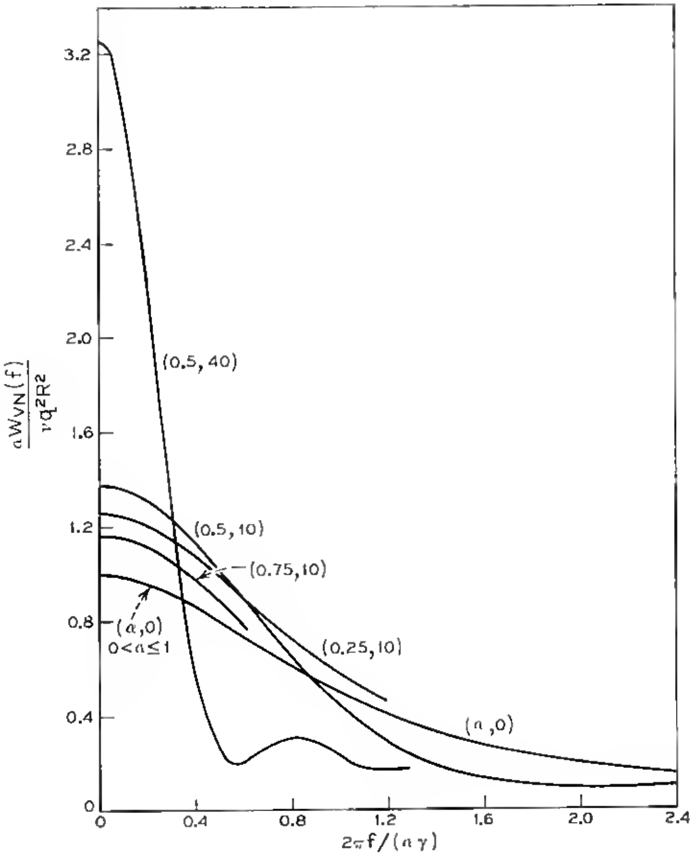

Fig. 3—Power spectrum of $V(t)$ in Fig. 1a.

$W_{V_N}(f)$ = 2-sided power spectrum of $V(t)$ minus DC spike due to $V_{dc} = \nu qR$.
$I(t) = \sum_k q\,\delta(t - t_k)$, ν = Arrival Rate, $\gamma = 1/(RC)$;
α = Fraction of time switch is closed;
T = Length of switch cycle;
$(\alpha, \gamma T)$ = Curve parameters.

We use the term "periodic" to mean "singly periodic." The more difficult case of "multiply periodic" variation is not considered. An example of the latter is given by the circuit of Fig. 1a in which the switch is operated by the function $f(t) = P \cos\ pt + Q \cos qt$, p and q being incommensurable. The switch is closed when $f(t) > 0$, and is open when $f(t) < 0$. Possibly such cases could be handled by the

method used by Bennett[5] to obtain the output of a rectifier when $P \cos pt + Q \cos qt$ is applied.

The (two-sided) power spectrum $W_V(f)$ [where $W_V(-f) = W_V(f)$] can be interpreted physically as follows. Let $V(t)$ be applied to an ideal filter which passes only the narrow band $f_1 < |f| < f_1 + \Delta f$, and let the filter be terminated in a resistance of one ohm. Then

$$[W_V(-f_1) + W_V(f_1)]\,\Delta f = 2W_V(f_1)\,\Delta f$$

is the time average of the power which would be dissipated in the one ohm resistance. The average must be taken over an interval long in comparison with $1/\Delta f$.

The analysis used here makes no attempt at mathematical rigor. Orders of summation and integration are interchanged freely, and assumptions are made which are physically plausible but which may be difficult to express in precise mathematical terms.

III. STATEMENT OF RESULTS FOR GENERAL SYSTEM

The results given in this section pertain to the general system shown in Fig. 2. The system is linear and is specified by the response $y(t) = h(t, \tau)$ to a unit impulse $x(t) = \delta(t - \tau)$ applied at time τ. The system varies periodically with period T so that

$$h(t + nT, \tau + nT) = h(t, \tau) \qquad n = \text{integer}. \tag{3}$$

In most of our work, the input $x(t)$ is the shot noise

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \delta(t - t_k) \tag{4}$$

where the random "arrival times" t_k occur at an average rate of ν/second and constitute a Poisson process. The impulse amplitudes a_k are independent random variables with

$$E(a_k) = E(a), \qquad E(a_k^2) = E(a^2). \tag{5}$$

Since the system is linear, the output corresponding to equation (4) is

$$y(t) = \sum_{k=-\infty}^{\infty} a_k h(t, t_k). \tag{6}$$

The function $h(t, \tau)$ is assumed to be such that the steps in the analysis are legitimate. In particular, it is assumed that when $0 \leq \tau \leq T$ and $|t| \to \infty$, $|h(t, \tau)|$ tends to 0 with sufficient rapidity to (*i*) make the various integrals converge, and (*ii*) ensure that the

times at which a long interval of operation begins and stops have no appreciable influence on the output during the major portion of the interval.

In Section VIII it is shown that $y(t)$ is the sum of a noise component $y_N(t)$ and a periodic (including dc) component $y_{per}(t)$:

$$y(t) = y_N(t) + y_{per}(t). \tag{7}$$

The power spectrum of $y_N(t)$ is

$$W_{yN}(f) = \frac{\nu E(a^2)}{T} \int_0^T |s(f, \tau)|^2 \, d\tau \tag{8}$$

where

$$s(f, \tau) = \int_{-\infty}^{\infty} e^{-i\omega t} h(t, \tau) \, dt, \qquad \omega = 2\pi f. \tag{9}$$

The periodic component of $y(t)$ is

$$\begin{aligned} y_{per}(t) &= \nu E(a) \sum_{m=-\infty}^{\infty} s_0(m/T) e^{i2\pi mt/T}, \\ &= \nu E(a) s_0(0) + 2\nu E(a) \text{ Real} \sum_{m=1}^{\infty} s_0(m/T) e^{i2\pi mt/T}, \end{aligned} \tag{10}$$

where

$$s_0(f) = \frac{1}{T} \int_0^T s(f, \tau) \, d\tau. \tag{11}$$

The dc part of $y(t)$ is given by the constant term in equation (10):

$$y_{dc} = \nu E(a) s_0(0). \tag{12}$$

Note that $y_{per}(t)$ is zero when $E(a)$ is zero.

The ensemble average $\langle y^l(t) \rangle$, which gives the lth moment of the distribution of the ensemble of $y(t)$'s at time t, is a periodic function of t of period T. For $l = 1$ and $l = 2$

$$\langle y(t) \rangle = \nu E(a) \sum_{n=-\infty}^{\infty} \int_0^T h(t + nT, \tau) \, d\tau, \tag{13}$$

$$\langle y^2(t) \rangle - \langle y(t) \rangle^2 = \nu E(a^2) \sum_{n=-\infty}^{\infty} \int_0^T h^2(t + nT, \tau) \, d\tau. \tag{14}$$

These equations give the first and second cumulants of the distribution of the $y(t)$'s. The lth cumulant at time t is

$$\kappa_l(t) = \nu E(a^l) \sum_{n=-\infty}^{\infty} \int_0^T h^l(t + nT, \tau) \, d\tau. \tag{15}$$

The periodic and noise components of $y(t)$ are related to the ensemble

averages by

$$y_{\text{per}}(t) = \langle y(t) \rangle = \kappa_1(t), \tag{16}$$

$$\langle y_N^2(t) \rangle = \langle y^2(t) \rangle - \langle y(t) \rangle^2 = \kappa_2(t). \tag{17}$$

The mean square value of $y_N^2(t)$, averaged over time, may be expressed in several ways:

$$\overline{y_N^2(t)} = \frac{1}{T} \int_0^T \langle y_N^2(t) \rangle \, dt = \frac{1}{T} \int_0^T \kappa_2(t) \, dt,$$

$$\begin{aligned} \overline{y_N^2(t)} &= \int_{-\infty}^{\infty} W_{yN}(f) \, df, \\ &= \frac{\nu E(a^2)}{T} \int_0^T d\tau \int_{-\infty}^{\infty} df \mid s(f, \tau) \mid^2, \\ &= \frac{\nu E(a^2)}{T} \int_0^T d\tau \int_{-\infty}^{\infty} dt \, h^2(t, \tau). \end{aligned} \tag{18}$$

All of the foregoing results pertain to the case in which the input $x(t)$ is the shot noise (4). Now let the input be zero-mean white gaussian noise with the power spectrum

$$W_x(f) = \begin{cases} N_0, & \mid f \mid < F; \\ 0, & \mid f \mid > F; \end{cases} \tag{19}$$

where $F \to \infty$. It is shown in Section IX that results for this input can be obtained from the preceding shot noise formulas by taking $a_k = \pm (N_0/\nu)^{\frac{1}{2}}$ with equal probability and letting $\nu \to \infty$. Then

$$\nu E(a) \to 0, \quad \nu E(a^2) \to N_0, \quad \text{and} \quad \nu E(a^l) \to 0 \quad \text{for} \quad l > 2. \tag{20}$$

Therefore $y_{\text{per}}(t) = \langle y(t) \rangle = 0$, and consequently $y(t)$ consists entirely of the noise component $y_N(t)$. Expressions for the output power spectrum $W_y(f)$ and the mean square values $\langle y^2(t) \rangle$, $\overline{y^2(t)}$ are obtained by replacing $\nu E(a^2)$ by N_0 in equations (8), (14), and (18):

$$\begin{aligned} W_y(f) &= \frac{N_0}{T} \int_0^T \mid s(f, \tau) \mid^2 d\tau, \\ \langle y^2(t) \rangle &= N_0 \sum_{n=-\infty}^{\infty} \int_0^T h^2(t + nT, \tau) \, d\tau, \\ \overline{y^2(t)} &= \frac{N_0}{T} \int_0^T d\tau \int_{-\infty}^{\infty} df \mid s(f, \tau) \mid^2, \\ &= \frac{N_0}{T} \int_0^T d\tau \int_{-\infty}^{\infty} dt \, h^2(t, \tau). \end{aligned} \tag{21}$$

In these expressions, $s(f, \tau)$ is still the Fourier transform (9) of $h(t, \tau)$. Equations (15) and (20) show that all of the cumulants except the second are zero. Therefore the ensemble of $y(t)$'s at time t is normally distributed about 0 with variance $\langle y^2(t) \rangle$ given by equation (21). The probability that $y(t)$ will lie between Y and $Y + dY$ at a time t picked at random is given by expression (113) in Section IX.

IV. RC CIRCUIT OF FIGURE 1a

In this section the results stated in Section III for general systems will be applied to the RC circuit shown in Fig. 1a. In this case the input $x(t)$ is the input $I(t)$ from the shot-noise current generator,

$$I(t) = \sum_{k=-\infty}^{\infty} q\delta(t - t_k) \tag{22}$$

where the individual charges (of q coulombs) arrive at an average rate of ν per second.

Comparison with the series (4) for $x(t)$ shows that $a_k = q$ and

$$E(a) = q, \qquad E(a^2) = q^2. \tag{23}$$

The output $V(t)$ is constant for intervals of length $(1 - \alpha)T$ while the switch is open. When the switch is closed $V(t)$ drifts either up or down, depending upon whether the input current is temporarily greater than or less than the leakage through R. The average value of $V(t)$ is $V_{dc} = \nu qR$ where νq is the average current flowing through R. It turns out that the mean square value of $V(t) - V_{dc}$ is $qV_{dc}/(2C)$. Furthermore, the circuit of Fig. 1a is unusual in that the distribution of the ensemble of $V(t)$'s at time t does not vary periodically with t.

Some insight into the behavior of the system can be obtained by considering the case when $T/\text{RC} \ll 1$. If the switch were closed all of the time ($\alpha = 1$), the usual shot effect formulas would hold and the two-sided power spectrum of $V(t)$ would be

$$\begin{aligned} W_V(f) &= \nu \left| \int_{-\infty}^{\infty} e^{-i\omega t} F(t)\, dt \right|^2 + V_{dc}^2\, \delta(f), \\ &= \frac{\nu q^2 R^2}{1 + (\omega\, \text{RC})^2} + V_{dc}^2\, \delta(f), \qquad \omega = 2\pi f, \end{aligned}$$

where $F(t)$ is the $V(t)$ due to a charge q arriving at time 0; $F(t) = (q/C) \exp(-t/\text{RC})$ for $t > 0$, and $F(t) = 0$ for $t < 0$. The first term in $W_V(f)$ is $W_{V_N}(f)$, the power spectrum of the noise component $V_N(t) =$

$V(t) - V_{dc}$, and the second term is the spike due to V_{dc}. Now, instead of $\alpha = 1$ let α be anywhere in $0 < \alpha < 1$, but take $T/\mathrm{RC} \ll 1$. The cycles are so brief that $V(t)$ does not change much during one cycle; and the situation is much like that for $\alpha = 1$ except that, in effect, ν is reduced to $\nu\alpha$, and $F(t)$ becomes $(q/C) \exp(-t\alpha/\mathrm{RC})$ because the condenser current flows only the fraction α of the time. Replacing ν by $\nu\alpha$ and $F(t)$ by its new expression leads to

$$W_{V_N}(f) \approx \frac{\nu q^2 R^2/\alpha}{1 + (\omega\, \mathrm{RC}/\alpha)^2}.$$

When T/RC is not small, the expressions for the power spectrum become much more complicated. We now turn to the general case in which T/RC and α are unrestricted except for $0 < \alpha < 1$.

The first step is to determine the response (the condenser voltage) $h(t, \tau)$ at time t to a *unit* impulse of current arriving at time τ where $0 < \tau < T$. When $\alpha T < \tau < T$, the impulse arrives when the switch is open, no charge reaches the condenser, no voltage appears across the condenser, and hence

$$h(t, \tau) \equiv 0 \quad \text{for all} \quad t \quad \text{when} \quad \alpha T < \tau < T. \tag{24}$$

When $0 < \tau < \alpha T$ the switch is closed, and the unit impulse of current arriving at time τ deposits a unit charge on the condenser. This charges the condenser to the voltage $1/C$. The voltage decreases exponentially as the charge leaks off through R until the switch opens at time αT. The voltage remains constant throughout the interval $\alpha T < t < T$ during which the switch is open. It resumes its exponential decay during $T < t < T + \alpha T$, remains constant during $T + \alpha T < t < 2T$, and so on. Hence when $0 < \tau < \alpha T$ the values of $h(t, \tau)$ are

$$\begin{array}{rl} 0, & -\infty < t < \tau; \\ C^{-1} \exp[-\gamma(t - \tau)], & \tau < t < \alpha T; \\ C^{-1} \exp[-\gamma(n\alpha T - \tau)], & (n-1)T + \alpha T < t < nT; \\ C^{-1} \exp[-\gamma(n\alpha T - \tau) - \gamma(t - nT)], & nT < t < nT + \alpha T; \end{array} \tag{25}$$

where $n = 1, 2, 3, \cdots$ and

$$\gamma = 1/(\mathrm{RC}). \tag{26}$$

Equation (6) for the output $y(t)$ becomes

$$V(t) = \sum_{k=-\infty}^{\infty} q h(t, t_k) \tag{27}$$

where $h(t, t_k)$ can be obtained from the relation $h(t + nT, \tau + nT) = h(t, \tau)$ and the values (24) and (25). From equation (9), the Fourier transform $s(f, \tau)$ of $h(t, \tau)$ is 0 when $\alpha T < \tau < T$ because, from (24), $h(t, \tau)$ is 0 in the same interval:

$$s(f, \tau) = 0, \qquad \alpha T < \tau < T. \tag{28}$$

For $0 < \tau < \alpha T$ we have, from equations (9) and (25),

$$\begin{aligned} s(f, \tau) &= \int_{-\infty}^{\infty} e^{-i\omega t} h(t, \tau)\, dt, \qquad \omega = 2\pi f; \\ &= \int_{\tau}^{\alpha T} C^{-1} \exp\left[-\gamma(t - \tau) - i\omega t\right] dt \\ &\quad + \sum_{n=1}^{\infty} C^{-1} \exp\left[-\gamma(n\alpha T - \tau)\right] \\ &\quad \times \left(\int_{(n-1)T+\alpha T}^{nT} e^{-i\omega t}\, dt + \int_{nT}^{nT+\alpha T} e^{-\gamma(t-nT)-i\omega t}\, dt \right). \end{aligned} \tag{29}$$

When the integrations are performed, the series summed, and the notation

$$z = e^{-i\omega T}, \qquad b = e^{-\gamma\alpha T} \tag{30}$$

introduced, some algebra carries equation (29) into

$$s(f, \tau) = \frac{C^{-1}e^{-i\omega\tau}}{\gamma + i\omega} + \frac{C^{-1}be^{\gamma\tau}}{1 - bz}(z - z^{\alpha})\left(\frac{1}{\gamma + i\omega} - \frac{1}{i\omega}\right) \tag{31}$$

for $0 < \tau < \alpha T$.

The integral (11) for $s_0(f)$ becomes

$$\begin{aligned} s_0(f) &= \frac{1}{T}\int_0^T s(f, \tau)\, d\tau, \qquad \omega = 2\pi f; \\ &= \frac{1}{T}\int_0^{\alpha T} s(f, \tau)\, d\tau. \end{aligned} \tag{32}$$

The function $s_0(f)$ is used solely to compute the periodic portion of the output, and therefore only the values of $s_0(m/T)$, where m is an integer, are of interest. For $f = m/T$ the value of $\omega = 2\pi f$ is $\omega = 2\pi m/T$, and $\omega T = 2\pi m$. Evaluation of the integral (32) for $s_0(f)$ leads to

$$\begin{aligned} s_0(0) &= 1/(C\gamma) = R, \\ s_0(m/T) &= 0, \qquad m \neq 0. \end{aligned} \tag{33}$$

As in equation (7), the output $V(t)$ can be expressed as the sum of a noise component and a periodic component,

$$V(t) = V_N(t) + V_{\text{per}}(t). \tag{34}$$

Since $s_0(m/T)$ is zero for $m \neq 0$, equation (10) shows that for Fig. 1a, the periodic component consists only of the dc component:

$$V_{\text{per}}(t) = V_{\text{dc}} = \nu E(a) s_0(0) = \nu q R. \tag{35}$$

The quantity νq is the average shot noise current (in amperes if q is measured in coulombs) flowing through R; and V_{dc} is the average IR drop across the resistance.

The value $\nu q R$ for $V_{\text{per}}(t)$ can also be obtained from equations (16) and (13),

$$\begin{aligned} V_{\text{per}}(t) &= \langle V(t) \rangle = \kappa_1(t), \\ &= \nu E(a) \int_0^T d\tau \sum_{n=-\infty}^{\infty} h(t + nT, \tau), \\ &= \nu E(a) C^{-1}/\gamma = \nu q C^{-1}/\gamma = \nu q R, \end{aligned} \tag{36}$$

where the expressions (24) and (25) for $h(t, \tau)$ are used in summing the series and evaluating the integral.

The values of the higher order cumulants $\kappa_l(t)$ follow almost immediately from equation (36). First observe that the expression (15) for $\kappa_l(t)$ can be obtained from the expression (13) for $\langle y(t) \rangle$ $(= \kappa_1(t))$ by replacing $E(a)$, $h(t + nT, \tau)$ by $E(a^l)$, $h^l(t + nT, \tau)$, respectively. Furthermore, $h^l(t + nT, \tau)$ can be obtained from $h(t + nT, \tau)$ by replacing C^{-1} and γ by C^{-l} and $l\gamma$, respectively. Therefore from equation (36),

$$\kappa_l(t) = \nu E(a^l) C^{-l}/(l\gamma) = \nu q^l C^{-l}/(l\gamma). \tag{37}$$

In particular, the variance of the distribution of the ensemble of $V(t)$'s at time t is

$$\langle V^2(t) \rangle - \langle V(t) \rangle^2 = \kappa_2(t) = \nu q^2 C^{-2}/(2\gamma) = \frac{q}{2C} V_{\text{dc}}. \tag{38}$$

The fact that this does not depend on t shows that the mean square value of the fluctuation about V_{dc} is also $qV_{\text{dc}}/(2C)$:

$$\overline{(V(t) - V_{\text{dc}})^2} = \overline{V_N^2(t)} = \frac{q}{2C} V_{dc} = \frac{\nu q^2 R}{2C}. \tag{39}$$

Equation (31) leads to the equation

$$\begin{aligned}\ln \varphi(z) &= \sum_{l=1}^{\infty} \kappa_l(t)(iz)^l/l!, \\ &= \frac{\nu}{\gamma} \sum_{l=1}^{\infty} \frac{(izq/C)^l}{l!\, l}, \\ &= \frac{\nu}{\gamma} \int_0^{zq/C} (e^{i\theta} - 1)\, d\theta/\theta,\end{aligned} \tag{40}$$

for the characteristic function $\varphi(z)$ of the distribution of the ensemble of $V(t)$'s at time t. The probability density of the distribution is given by

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(z) e^{-iVz}\, dz \tag{41}$$

where $\ln \varphi(z)$ can be expressed in terms of sine and cosine integrals. The integral (41) also gives the probability density of the value of a particular member of the ensemble at a time selected at random.

The power spectrum $W_{V_N}(f)$ of $V_N(t) = V(t) - V_{dc}$ is obtained by substituting the value (31) of $s(f, \tau)$ in (8).

$$\begin{aligned}W_{V_N}(f) &= \frac{\nu q^2}{T} \int_0^{\alpha T} |s(f, \tau)|^2\, d\tau, \\ &= \frac{\nu q^2 R^2}{1 + (\omega/\gamma)^2} \operatorname{Real} \left[\alpha + \frac{(1 - bz^{\alpha})(1 - z^{1-\alpha})\gamma(\gamma - i\omega)}{T(1 - bz)\omega^2(\gamma + i\omega)} \right],\end{aligned} \tag{42}$$

where $\omega = 2\pi f$, $\gamma = 1/\mathrm{RC}$, and z and b are given by (30):

$$z = e^{-i\omega T}, \qquad b = e^{-\gamma \alpha T}.$$

An outline of the evaluation of the integral is given in Appendix A. The curves plotted in Fig. 3 were computed from equation (42). It can be shown that

$$\begin{aligned}&W_{V_N}(0) = \nu q^2 R^2 [2 - \alpha + \tfrac{1}{2}\gamma T(1 - \alpha)^2 (1 + b)(1 - b)^{-1}]; \\ &W_{V_N}(f) \to \nu q^2 R^2 \gamma^2 \alpha/\omega^2, \quad \text{as} \quad f \to \infty; \\ &W_{V_N}(f) \to \frac{\nu q^2 R^2/\alpha}{1 + [\omega/(\gamma\alpha)]^2}, \quad \text{as} \quad T \to 0.\end{aligned} \tag{43}$$

In Fig. 3, the quantity $\alpha W_{V_N}(f)/(\nu q^2 R^2)$ is plotted as a function of $\omega/(\gamma\alpha) = \omega\ \mathrm{RC}/\alpha = 2\pi f\ \mathrm{RC}/\alpha$. The parameters are α and $\gamma T = T/(\mathrm{RC})$. These coordinates were chosen because tbe exact computations made from equation (42) give nearly the same values as does the last approximation ($T \to 0$) in (43) for values of γT less tban, say,

5.0. From

$$\int_0^\infty W_{VN}(f)\,df = \tfrac{1}{2}\overline{V^2(t)} = \frac{1}{2}\frac{\nu q^2 R}{2C}$$

it can be shown that the area under any curve in Fig. 3 is $\pi/2$. As $\gamma T \to \infty$, the ordinate at $f = 0$ ultimately increases as

$$\alpha(2 - \alpha) + \frac{\alpha\gamma T}{2}(1 - \alpha)^2$$

which, for γT fixed, has a maximum at

$$\alpha = \frac{1}{3} + \frac{4}{3\gamma T}.$$

The oscillations in the curves for the large values of γT can be correlated with the oscillations in a $(\sin f/f)^2$ type of spectrum associated with the flat portions of length $(1 - \alpha)T$ in $V(t)$.

When the shot noise current generator in Fig. 1a is replaced by a zero-mean white noise current generator with a flat, two-sided power spectrum $W_I(f) = N_{I0}$, the dc component of $V(t)$ becomes 0 and $V(t)$ is distributed normally about zero with variance

$$\langle V^2(t)\rangle = \overline{V^2(t)} = N_{I0}/(2\gamma C^2). \tag{44}$$

This $V(t)$ is an example of a stochastic process in which the distribution of the ensemble at time t is normal and does not change with t, but the process is still not a stationary gaussian process because $dV(t)/dt$ is zero during the intervals that the switch is open.

The power spectrum $W_V(f)$ is given by equations (42) and (43) with the multiplier νq^2 replaced by N_{I0}. In the particular case in which the period T is small compared to the time constant RC, the last approximation given in equation (43) goes into

$$W_V(f) \to \frac{N_{I0}R^2/\alpha}{1 + (\omega\,\mathrm{RC}/\alpha)^2}. \tag{45}$$

The Princeton Applied Research notes[3] obtained by Sell give results associated with this approximation.

By Thevenin's theorem, the portion of Fig. 1a consisting of the infinite impedance shot noise current generator plus the resistance R shunting the generator can be replaced by a zero impedance shot noise voltage generator in series with R. The currents and voltages in the remaining portion of the circuit are unchanged by this replacement. The voltage of the new generator is $V_g(t) = I(t)R$; and its two-sided power spectrum $W_{Vg}(f)$ is flat and equal to $N_{V0} =$

$N_{I0}R^2$. The statistical results for the voltage $V(t)$ across C can be expressed in terms of N_{V0} by replacing N_{I0} by N_{V0}/R^2 (that is, νq^2 by N_{V0}/R^2). For example, equation (44) becomes

$$\overline{V^2}(t) = N_{V0}/(R^2 2\gamma C^2) = N_{V0}/(2RC). \tag{46}$$

V. RC CIRCUIT OF FIGURE $1b$

The input shot noise current $I(t)$ in Fig. 1b is the same as in Fig. 1a, and is given by the sum (22) of impulses of weight q. The switch is in position a during the first part of the cycle, $nT < t < nT + \alpha T$; and in position b during the second part, $nT + \alpha T < t < (N + 1)T$.

The condenser voltage $V(t)$ increases more often than not during the first part of the cycle. It always decreases during the second part. Unlike the circuit Fig. 1a, $V(t)$ has a periodic portion $V_{\text{per}}(t)$ which includes variable terms in addition to V_{dc}.

Just as in Fig. 1a, we have $a_k = q$ and $E(a) = q$, $E(a^2) = q^2$ .The response $h(t, \tau)$ at time t to a unit impulse of current arriving at τ, where $0 < \tau < T$, is

$$\begin{aligned} &0 \quad \text{for} \quad -\infty < t < \tau, \\ C^{-1} \exp [-\gamma(t - \tau)] \quad &\text{for} \quad 0 < \tau < \alpha T \quad \text{and} \quad \tau < t, \\ &0 \quad \text{for} \quad \alpha T < \tau < T \quad \text{and all} \quad t. \end{aligned} \tag{47}$$

As before, $\gamma = 1/(RC)$ and

$$V(t) = \sum_{k=-\infty}^{\infty} qh(t, t_k). \tag{48}$$

The Fourier transform of $h(t, \tau)$ is

$$\begin{aligned} s(f, \tau) &= \int_{\tau}^{\infty} e^{-i\omega t} C^{-1} e^{-\gamma(t-\tau)}\, dt, \\ &= \frac{C^{-1}e^{-i\omega\tau}}{\gamma + i\omega}, \qquad \omega = 2\pi f; \end{aligned} \tag{49}$$

for $0 < \tau < \alpha T$, and $s(f, \tau) = 0$ for $\alpha T < \tau < T$. The integral $s_0(f)$ used in computing $V_{\text{per}}(t)$ in $V(t) = V_N(t) + V_{\text{per}}(t)$ is

$$\begin{aligned} s_0(f) &= \frac{1}{T} \int_0^T s(f, \tau)\, d\tau, \\ &= (1 - e^{-i\omega\alpha T})/[i\omega TC(\gamma + i\omega)], \\ s_0(0) &= \alpha/C\gamma = \alpha R. \end{aligned} \tag{50}$$

The dc portion of $V_{\text{per}}(t)$ is

$$V_{\text{dc}} = \nu E(a) s_0(0) = \nu q \alpha R \tag{51}$$

and, from the general expression (10) for $V_{\text{per}}(t)$,

$$V_{\text{per}}(t) = V_{\text{dc}} + 2 \text{ Real} \sum_{m=1}^{\infty} \left[\frac{V_{\text{dc}}}{1 + i(\omega/\gamma)} \left(\frac{1 - e^{-i\omega\alpha T}}{i\omega\alpha T} \right) e^{i\omega t} \right]_{\omega = 2\pi m/T}. \tag{52}$$

By working with

$$V_{\text{per}}(t) = \langle V(t) \rangle = \nu q \sum_{n=-\infty}^{\infty} \int_0^T h(t + nT, \tau)\, d\tau \tag{53}$$

it can be shown that $V_{\text{per}}(t)$ increases from A exp $(-\gamma T)$ at $t = 0$ to A exp $(-\gamma \alpha T)$ at $t = \alpha T$, and then decreases to A exp $(-\gamma T)$ at $t = T$ and so on, where

$$A = \frac{V_{\text{dc}}}{\alpha} \frac{e^{\gamma \alpha T} - 1}{1 - e^{-\gamma T}}. \tag{54}$$

The power spectrum $W_{VN}(f)$ of the noise portion $V_N(t)$ of $V(t)$ is given by equation (8) and the expression (49) for $s(f, \tau)$.

$$\begin{aligned} W_{VN}(f) &= \frac{\nu E(a^2)}{T} \int_0^T |s(f, \tau)|^2\, d\tau, \\ &= \frac{\nu q^2 C^{-2}}{T} \int_0^{\alpha T} \frac{d\tau}{\gamma^2 + \omega^2}, \qquad \omega = 2\pi f; \\ &= \frac{\nu q^2 C^{-2} \alpha}{\gamma^2 + \omega^2} = R q V_{\text{dc}} / [1 + (\omega \text{ RC})^2]. \end{aligned} \tag{55}$$

Integrating $W_{VN}(f)$ from $f = -\infty$ to $f = +\infty$ shows that the time average of $V_N^2(t)$ is

$$\overline{V_N^2(t)} = \frac{q}{2C} V_{\text{dc}} \tag{56}$$

just as in the case of Fig. 1a [see equation (39)]. However, in Fig. 1a, $V_{\text{dc}} = \nu q R$; whereas in Fig. 1b, $V_{\text{dc}} = \nu q \alpha R$.

When the shot noise current generator in Fig. 1b is replaced by a zero-mean white noise current generator with flat power spectrum $W_I(f) = N_{I0}$, the periodic component $V_{\text{per}}(t)$ vanishes and the power spectrum of $V(t)$ is obtained by replacing νq^2 in equation (55) by N_{I0}:

$$W_V(f) = W_{VN}(f) = N_{I0} \frac{C^{-2} \alpha}{\gamma^2 + \omega^2}, \qquad \omega = 2\pi f. \tag{57}$$

The time average of $V^2(t)$ obtained by integrating equation (57) is

$$\overline{V^2(t)} = N_{I0} \frac{\alpha R}{2C}. \tag{58}$$

Although the periodic component $V_{\text{per}}(t) = \langle V(t) \rangle$ is zero, the ensemble variance $\langle V^2(t) \rangle$ at time t is a periodic function of t. It may be calculated from the second of equations (21) in which $h^2(t, \tau)$ is obtained by squaring the expressions (47) for $h(t, \tau)$.

VI. THE ENSEMBLE AVERAGE $\langle y(t) \rangle$

In this section and the two following ones, the arguments used to deal with shot noise will be used to determine the power spectrum and the moments (more precisely, the cumulants) of the distribution of the output $y(t)$ of the periodically varying system shown in Fig. 2. The input $x(t)$ is taken to be shot noise consisting of a train of randomly arriving impulses.

Let the system of Fig. 2 start operating at time $t = 0$ and run to $t = T_1$ where $T_1 = NT$ with $N \gg 1$. Let the number of impulses arriving in $0 < t < T_1$ be the random variable K, and let the input be

$$\begin{aligned} x(t) &= \sum_{k=1}^{K} a_k \delta(t - t_k), \qquad && K \geqq 1; \\ x(t) &= 0, && K = 0; \end{aligned} \tag{59}$$

where, as in equation (4), the impulse amplitudes a_k are independent random variables with probability density $q(a)$ and expected value

$$E(a_k) = E(a), \qquad E(a_k^2) = E(a^2). \tag{60}$$

The arrival times $t_1, t_2, \cdots t_k$ are independent random variables with

$$\text{Prob}\,[t < t_k < t + dt] = dt/T_1. \tag{61}$$

The number of arrivals K has the Poisson distribution

$$\begin{aligned} \text{Prob}\,[K = L] &= (\nu T_1)^L e^{-\nu T_1}/L!, \\ E(K) &= \nu T_1, \\ E(K^2 - K) &= (\nu T_1)^2, \end{aligned} \tag{62}$$

where ν is the expected number of arrivals per second.

The output produced by the input (59) is

$$\begin{aligned} y(t) &= \sum_{k=1}^{K} a_k h(t, t_k), \qquad && K \geqq 1; \\ y(t) &= 0, && K = 0. \end{aligned} \tag{63}$$

When t is fixed, $y(t)$ may be regarded as a random variable since it depends on the random variables K, a_k, t_k. The lth moment of the distribution of $y(t)$ is the ensemble average $\langle y^l(t)\rangle$. Usually $\langle y^l(t)\rangle$ will depend on t and be periodic with period T. We shall be concerned with the first moment $\langle y(t)\rangle$ in the remainder of this section.

When the right side of the first part of equation (63) is averaged over the ensemble of a_k's, it becomes

$$E(a) \sum_{k=1}^{K} h(t, t_k), \qquad K \geqq 1. \tag{64}$$

Averaging this over the ensemble of t_k's gives

$$E(a) \sum_{k=1}^{K} \frac{1}{T_1} \int_0^{T_1} dt_k\, h(t, t_k) = KE(a) \frac{1}{T_1} \int_0^{T_1} dt_k\, h(t, t_k) \tag{65}$$

where use has been made of the fact that all of the terms in the series on the left are equal. Finally, averaging over the ensemble of K's with the help of $E(K) = \nu T_1$ gives

$$\langle y(t)\rangle = \nu E(a) \int_0^{T_1} dt_k\, h(t, t_k). \tag{66}$$

Dividing the interval $(0, T_1)$ into N equal intervals of length T, setting $t_k = nT + \tau$, and using the periodic property $h(t + nT, \tau + nT) = h(t, \tau)$ leads to

$$\begin{aligned}
\langle y(t)\rangle &= \nu E(a) \sum_{n=0}^{N-1} \int_{nT}^{(n+1)T} dt_k\, h(t, t_k), \\
&= \nu E(a) \sum_{n=0}^{N-1} \int_0^{T} d\tau\, h(t - nT + nT, nT + \tau), \\
&= \nu E(a) \sum_{n=0}^{N-1} \int_0^{T} d\tau\, h(t - nT, \tau).
\end{aligned} \tag{67}$$

Equation (67) holds when the system starts operating at $t = 0$ and stops at $t = T_1$. The following heuristic argument suggests that when (*i*) the system runs from $t = -\infty$ to $+\infty$, and (*ii*) $h(t, \tau)$ is such that only recent arrivals are of importance in determining the present state of the system, the analogue of equation (67) is

$$\langle y(t)\rangle = \nu E(a) \sum_{n=-\infty}^{\infty} \int_0^{T} d\tau\, h(t - nT, \tau). \tag{68}$$

We assume that, for $0 < \tau < T$, $h(u, \tau)$ becomes negligible when $|u| \geq mT$ where m is a small integer. We define t to be in the "in-

terior" of $(0, T_1)$ when

$$mT < t < T_1 - mT.$$

If t is in the interior of $(0, T_1)$, the summation in equation (67) can be written as

$$\sum_{n=0}^{N-1} = \sum_{|t-nT|<mT} = \sum_{n=-\infty}^{\infty}$$

because $h(t - nT, \tau)$ is negligible except when $|t - nT| < mT$. Hence when t is in the interior of $(0, T_1)$ and the system runs from 0 to T_1 $< y(t) >$ is given by both (67) and (68).

In the interior of $(0, T_1)$ the starting and stopping transients near 0 and T_1 have died out, and $y(t)$ is the same irrespective of whether the system runs from 0 to T_1 or from $-\infty$ to $+\infty$. Hence when t is in the interior of $(0, T_1)$ and the system runs from $-\infty$ to $+\infty$, $\langle y(t) \rangle$ is again given by both (67) and (68).

The right side of equation (68) is a periodic function of t of period T. Physical considerations suggest that when the system runs from $-\infty$ to $+\infty$ $\langle y(t) \rangle$ is also a periodic function of period T. Since $\langle y(t) \rangle$ and the right side of equation (68) are equal when t lies in the interior of $(0, T_1)$ (which extends over more than one period), it is plausible to say that the equality holds for all values of t. This is what we wished to show.

Equation (68) appears as equation (13) in Section III. The sign of the index of summation n has been changed to make it easier to apply the formula.

VII. THE CUMULANTS FOR $y(t)$

The lth moment $\langle y^l(t) \rangle$ may be expressed in terms of the first l cumulants $\kappa_1(t), \cdots \kappa_l(t)$ of the distribution and conversely. For $l = 1$ and $l = 2$.

$$\begin{aligned} \kappa_1(t) &= \langle y(t) \rangle, \\ \kappa_2(t) &= \langle y^2(t) \rangle - \langle y(t) \rangle^2. \end{aligned} \tag{69}$$

The cumulants are defined by

$$\ln \varphi(z) = \sum_{l=1}^{\infty} \kappa_l(t)(iz)^l/l! \tag{70}$$

where $\varphi(z)$ is the characteristic function

$$\varphi(z) = \langle \exp [izy(t)] \rangle. \tag{71}$$

The method of averaging over the ensemble used in the preceding section to obtain $\langle y(t) \rangle$ will now be applied to calculate $\langle \exp [izy(t)] \rangle$. We have, because of the independence of the a_k's and t_k's,

$$\begin{aligned} \langle \exp [izy(t)] \rangle &= \left\langle \exp \left[iz \sum_{k=1}^{K} a_k h(t, t_k) \right] \right\rangle, \\ &= \sum_{K=0}^{\infty} \frac{(\nu T_1)^K}{K!} e^{-\nu T_1} \langle \exp [iza_k h(t, t_k)] \rangle^K, \\ &= \exp [-\nu T_1 + \nu T_1 \langle \exp [iza_k h(t, t_k)] \rangle]. \end{aligned} \tag{72}$$

Therefore, upon using the definition (71) of $\varphi(z)$ and the probability densities of a_k and t_k,

$$\ln \varphi(z) = -\nu T_1 + \nu T_1 \int_{-\infty}^{\infty} da_k \, q(a_k) \int_0^{T_1} \frac{dt_k}{T_1} \exp [iza_k h(t, t_k)]. \tag{73}$$

Expanding both sides in powers of z and equating coefficients of $(iz)^l/l!$,

$$\begin{aligned} \kappa_l(t) &= \nu \int_{-\infty}^{\infty} da_k \, q(a_k) a_k^l \int_0^{T_1} dt_k \, h^l(t, t_k), \\ &= \nu E(a^l) \int_0^{T_1} dt_k \, h^l(t, t_k). \end{aligned} \tag{74}$$

When $l = 1$, equation (74) reduces to equation (66) for $\langle y(t) \rangle$. The steps that lead from equation (66) to the final expression for $\langle y(t) \rangle$ carry (74) into

$$\kappa_l(t) = \nu E(a^l) \sum_{n=-\infty}^{\infty} \int_0^{T} d\tau \, h^l(t - nT, \tau). \tag{75}$$

This appears as equation (15) in Section III with n replaced by $-n$.

VIII. THE POWER SPECTRUM OF $y(t)$

When $h(t, \tau)$ is such that $y(t)$ has the two-sided power spectrum $W_y(f)$, it is given by[4]

$$W_y(f) = \operatorname*{limit}_{T_1 \to \infty} \langle | S(f, T_1) |^2 \rangle / T_1 \tag{76}$$

where

$$
\begin{aligned}
S(f, T_1) &\equiv S(f, T_1; K; a_1, \cdots, a_K; t_1, \cdots, t_K) \\
&= \int_0^{T_1} dt\, e^{-i\omega t} y(t), \qquad \omega = 2\pi f; \\
&= \sum_{k=1}^{K} a_k \int_0^{T_1} dt\, e^{-i\omega t} h(t, t_k); \\
&= \sum_{k=1}^{K} a_k \int_{-t_k}^{T_1 - t_k} du\, e^{-i\omega(t_k+u)} h(t_k + u, t_k).
\end{aligned} \tag{77}
$$

In the derivation of equation (68) for $\langle y(t) \rangle$, the limits of summation $n = 0$, $n = N-1$ were replaced by $n = -\infty$, $n = \infty$. In much the same way, we assume that in (77) the limits of integration $-t_k$, $T_1 - t_k$ can be replaced by $-\infty$, $+\infty$ in all but a negligible fraction of the terms (those with t_k near 0 or T_1). This presupposes a sufficiently rapid decrease in the value of $|h(t_k + u, t_k)|$ as $|u| \to \infty$. Heuristically, we picture $h(t_k + u, t_k)$ as being negligible except when u is small. When T_1 is very large, most of the t_k's and $(T_1 - t_k)$'s will be large. Consequently, for most of the t_k's, $h(t_k + u, t_k)$ will be negligible when u is less than $-t_k$ or greater than $T - t_k$.

This assumption allows us to replace equations (76) and (77) by

$$
W_y(f) = \lim_{T_1 \to \infty} \langle | S_a(f, T_1) |^2 \rangle / T_1 \tag{78}
$$

and

$$
\begin{aligned}
S_a(f, T_1) &= \sum_{k=1}^{K} a_k \int_{-\infty}^{\infty} du\, e^{-i\omega(t_k+u)} h(t_k + u, t_k), \\
&= \sum_{k=1}^{K} a_k s(f, t_k),
\end{aligned} \tag{79}
$$

where

$$
s(f, \tau) = \int_{-\infty}^{\infty} dt\, e^{-i\omega t} h(t, \tau). \tag{80}
$$

From equation (79)

$$
\begin{aligned}
| S_a(f, T_1) |^2 &= S_a(f, T_1) S_a^*(f, T_1), \\
&= \sum_{k=1}^{K} \sum_{l=1}^{K} a_k a_l s(f, t_k) s^*(f, t_l),
\end{aligned} \tag{81}
$$

where the star denotes conjugate complex. The terms in equation (81) can be divided into two types. For Type I, $l = k$, and for Type II, $l \neq k$. It is convenient to take their ensemble averages separately.

The typical Type I term is

$$a_k^2 \mid s(f, t_k) \mid^2. \tag{82}$$

There are K terms of Type I in the double sum (81), and all of them are of the form (82). Therefore, when use is made of $E(K) = \nu T_1$, the contribution of the Type I terms to $\langle |S_a(f, T_1)|^2 \rangle$ is found to be

$$\nu E(a^2) \int_0^{T_1} d\tau \mid s(f, \tau) \mid^2. \tag{83}$$

The typical Type II term in (86) is

$$a_k a_l s(f, t_k) s^*(f, t_l), \qquad l \neq k.$$

When averaged with respect to a_k, a_l, t_k, t_l it becomes

$$\left| E(a) \int_0^{T_1} \frac{dt_k}{T_1} s(f, t_k) \right|^2. \tag{84}$$

There are $K^2 - K$ terms of Type II in the double sum (81) and all of them have the average value (84). Therefore, when use is made of $E(K^2 - K) = \nu^2 T_1^2$, the contribution of Type II terms to $\langle \mid S_a(f, T_1) \mid^2 \rangle$ is found to be

$$\left| \nu E(a) \int_0^{T_1} d\tau \, s(f, \tau) \right|^2. \tag{85}$$

Adding the contributions of Type I and Type II, and inserting the resulting expression for $\langle |S_a(f, T_1)|^2 \rangle$ in equation (78) for the power spectrum gives, with $\omega = 2\pi f$ and $s(f, \tau)$ given by (80),

$$W_y(f) = \lim_{T_1 \to \infty} \frac{1}{T_1} \left[\nu E(a^2) \int_0^{T_1} d\tau \mid s(f, \tau) \mid^2 + \left| \nu E(a) \int_0^{T_1} d\tau \, s(f, \tau) \right|^2 \right] \tag{86}$$

provided $s(f, \tau)$ [i.e., $h(t, \tau)$] is such that the limit exists. If, for certain frequencies, the function of T_1 following the limit sign ultimately increases linearly with T_1, $W_y(f)$ has an infinite spike at these frequencies. This means that $y(t)$ has sinusoidal components at these frequencies.

So far in this section, the time variation of the system has not been assumed to be periodic. Now we apply (86) to the case in which the system varies periodically with period T and, in accordance with equations (3) and (80),

$$\begin{aligned} h(t - nT + nT, \tau + nT) &= h(t - nT, \tau), \\ s(f, \tau + nT) &= e^{-i\omega nT} s(f, \tau). \end{aligned} \tag{87}$$

In (86) set $T_1 = NT$ and let $N \to \infty$. Then

$$\frac{1}{T_1}\int_0^{T_1} d\tau \mid s(f,\tau) \mid^2 = \frac{1}{T}\int_0^{T} d\tau \mid s(f,\tau) \mid^2,$$

$$\int_0^{T_1} d\tau\, s(f,\tau) = T \sum_{n=0}^{N-1} e^{-i\omega nT} s_0(f), \tag{88}$$

$$= Ts_0(f)\,\frac{1 - e^{-i\omega NT}}{1 - e^{-i\omega T}},$$

where

$$s_0(f) = \frac{1}{T}\int_0^T d\tau\, s(f,\tau), \qquad \omega = 2\pi f. \tag{89}$$

The contribution of the second term in (86) contains the factor

$$\begin{aligned} \operatorname*{limit}_{N\to\infty} \frac{T}{N}\left|\frac{1 - e^{-i\omega NT}}{1 - e^{-i\omega T}}\right|^2 &= \operatorname*{limit}_{N\to\infty} \frac{T}{N}\left|\frac{\sin\dfrac{\omega NT}{2}}{\sin\dfrac{\omega T}{2}}\right|^2 \\ &= \sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{T}\right), \end{aligned} \tag{90}$$

where the last step follows from the relations used in the proof of Fejér's theorem in the theory of Fourier series. When these results are used in equation (86), it goes into

$$\begin{aligned} W_y(f) &= \nu E(a^2)\,\frac{1}{T}\int_0^T d\tau \mid s(f,\tau) \mid^2 \\ &\quad + \nu^2 E^2(a) \sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{T}\right) \left| s_0\left(\frac{m}{T}\right) \right|^2. \end{aligned} \tag{91}$$

Equation (91) shows spikes in $W_y(f)$ at $f = m/T$ where m is an integer. The spike at $f = 0$ corresponds to the dc component y_{dc} of $y(t)$, and the spikes at $\pm m/T$ to the sinusoidal component

$$A_m \cos\,[2\pi m(t/T) + \theta_m] \tag{92}$$

in $y(t)$. The expression (91) for $W_y(f)$ shows that the (time) average powers in these components are

$$\begin{aligned} y_{dc}^2 &= [\nu E(a) s_0(0)]^2, \\ \tfrac{1}{2}A_m^2 &= [\nu E(a)]^2[\mid s_0(-m/T) \mid^2 + \mid s_0(m/T) \mid^2], \\ &= 2[\nu E(a) \mid s_0(m/T) \mid]^2. \end{aligned} \tag{93}$$

Equations (93) tell us nothing about the sign of y_{dc} or about the phase angle θ_m. One of the several ways to get this information is to imagine $y(t)$ expanded in a Fourier series of long period T_1,

$$\begin{aligned}
y(t) &= \sum_{-\infty}^{\infty} c_n \exp(i2\pi nt/T_1),\\
c_n &= \frac{1}{T_1}\int_0^{T_1} dt\, y(t) \exp(-i2\pi nt/T_1),\\
&= \frac{1}{T_1} S\left(\frac{n}{T_1}, T_1\right), \qquad (94)\\
&\approx \frac{1}{T_1} S_a\left(\frac{n}{T_1}, T_1\right),\\
&= \frac{1}{T_1}\sum_{k=1}^{K} a_k s\left(\frac{n}{T_1}, t_k\right),
\end{aligned}$$

where we have used equations (77) and (79) for $S(f, T_1)$ and its approximation $S_a(f, T_1)$. The expression (91) for $W_y(f)$ shows that the c_n's may be divided into two classes; those corresponding to the frequencies $n/T_1 = m/T$, i.e., $n = mN$ (discrete sinusoidal components) and those corresponding to $n \neq mN$ (noise). For the first class, c_n is $0(1)$ and nearly the same for most $y(t)$'s of the ensemble. For the second class, c_n is $0(T_1^{-\frac{1}{2}})$ and varies greatly from member to member.

To obtain the discrete sinusoidal component in $y(t)$ of frequency m/T, we set $n = mN$ in equation (94) and apply the procedure used in Section VI (to obtain $\langle y(t)\rangle$) to average c_n over the ensemble.

$$\begin{aligned}
\langle c_n\rangle_{n=mN} &= \frac{1}{T_1} E(K)E(a)\int_0^{T_1}\frac{dt_k}{T_1} s\left(\frac{m}{T}, t_k\right),\\
&= \nu E(a)T\sum_{n=0}^{N-1}\frac{1}{T_1}\exp\left[\frac{-i2\pi m(nT)}{T}\right] s_0\left(\frac{m}{T}\right), \qquad (95)\\
&= \nu E(a)s_0(m/T),
\end{aligned}$$

where we have used equations (88) and (89) with $\omega = 2\pi m/T$. We therefore write $y(t)$ as the sum of a noise component $y_N(t)$, consisting of the sum of terms of the second class, and a periodic component $y_{per}(t)$, consisting of the sum of terms of the first class:

$$y(t) = y_N(t) + y_{per}(t). \qquad (96)$$

The power spectrum of $y_N(t)$ is the first term in the expression (91) for $W_y(t)$:

$$W_{yN}(f) = \nu E(a^2) \frac{1}{T} \int_0^T d\tau \mid s(f, \tau) \mid^2. \tag{97}$$

The periodic component is, from (95),

$$y_{\text{per}}(t) = \nu E(a) \sum_{m=-\infty}^{\infty} s_0(m/T) \exp [i2\pi mt/T]. \tag{98}$$

The parts $y_N(t)$ and $y_{\text{per}}(t)$ of $y(t)$ are related to the ensemble averages by

$$y_{\text{per}}(t) = \langle y(t) \rangle = \kappa_1(t), \tag{99}$$

$$\langle y_N^2(t) \rangle = \langle y^2(t) \rangle - \langle y(t) \rangle^2 = \kappa_2(t). \tag{100}$$

Equation (99) can be proved by showing that the mth Fourier coefficients of $y_{\text{per}}(t)$ and $\langle y(t) \rangle$ are equal for all integers m, i.e., by showing that

$$\nu E(a) s_0(m/T) = \frac{1}{T} \int_0^T \langle y(t) \rangle \exp (-i2\pi mt/T)\, dt. \tag{101}$$

When the series (68) for $\langle y(t) \rangle$ is substituted on the right, the summation and the integration with respect to t from 0 to T combine to give an integral in t with limits $\pm\infty$. This integral can be evaluated with the help of the integral (80) for $s(f, \tau)$ and leads to the verification of (99). Equation (100) follows from the ensemble average of the square of

$$y_N(t) = y(t) - y_{\text{per}}(t) = y(t) - \langle y(t) \rangle.$$

Setting $l = 2$ in the expression (75) for $\kappa_l(t)$ and using (100) gives an expression for the ensemble average of $y_N^2(t)$ at time t,

$$\langle y_N^2(t) \rangle = \kappa_2(t) = \nu E(a^2) \sum_{n=-\infty}^{\infty} \int_0^T h^2(t - nT, \tau)\, d\tau. \tag{102}$$

It follows from (102) that when the variance $\langle y_N^2(t) \rangle$ varies with t, it varies periodically with period T. When equation (102) is averaged over a period and use is made of the ergodic relation (2), we get the time average

$$\overline{y_N^2(t)} = \overline{\langle y_N^2(t) \rangle} = \frac{\nu E(a^2)}{T} \int_0^T d\tau \int_{-\infty}^{\infty} dt\, h^2(t, \tau). \tag{103}$$

From the expression (97) for $W_{yN}(f)$, we get a second expression for $\overline{y_N^2(t)}$

$$\overline{y_N^2(t)} = \int_{-\infty}^{\infty} W_{yN}(f)\, df = \frac{\nu E(a^2)}{T} \int_0^T d\tau \int_{-\infty}^{\infty} df \mid s(f, \tau) \mid^2. \tag{104}$$

The equality of (103) and (104) can also be proved directly by using the Fourier integral (80) relating $s(f, \tau)$ and $h(t, \tau)$.

IX. WHITE GAUSSIAN NOISE INPUT

Let the input $x(t)$ of the periodically varying system shown in Fig. 2 be white gaussian noise with zero mean. Here we show that the output $y(t)$ has no dc or sinusoidal components, and that the power spectrum of $y(t)$ is

$$W_y(f) = \frac{N_0}{T} \int_0^T |\, s(f, \tau)\,|^2 \, d\tau \tag{105}$$

where the power spectrum of $x(t)$ is $W_x(f) = N_0$ for $|f| < F$ and $W_x(f) = 0$ for $|f| > F$ with $F \to \infty$.

Consider Fig. 4 in which an ideal low pass filter which passes only the frequencies $|f| < F$ has been inserted between the input and the periodically varying network specified by $h(t, \tau)$.

When $x(t)$ is a unit impulse applied at time τ, $x(t) = \delta(t - \tau)$, the filter output at time $t = t_1$ is

$$z(t_1) = \frac{\sin 2\pi F(t_1 - \tau)}{\pi(t_1 - \tau)}, \tag{106}$$

and the system output at time t is

$$y(t) = \int_{-\infty}^{\infty} h(t, t_1) \frac{\sin 2\pi F(t_1 - \tau)}{\pi(t_1 - \tau)} \, dt_1 . \tag{107}$$

Thus, when $h(t, t_1)$ satisfies conditions associated with the Fourier integral theorem, $y(t)$ tends to $h(t, \tau)$ as $F \to \infty$; a result which follows immediately from physical considerations.

Take $x(t)$ to be the shot noise given by (4) in which, for given values of N_0 and ν, $a_k = \pm(N_0/\nu)^{\frac{1}{2}}$ with equal probability. Then

$$\begin{aligned} \nu E(a) &= \nu E(a_k) = 0, \\ \nu E(a^2) &= \nu E(a_k^2) = N_0 , \end{aligned} \tag{108}$$

and the filter output is the zero-mean shot noise

$$z(t) = \sum_{-\infty}^{\infty} a_k \frac{\sin 2\pi F(t - t_k)}{\pi(t - t_k)} \tag{109}$$

with the power spectrum

$$\begin{aligned} W_z(f) &= \nu E(a^2) \left| \int_{-\infty}^{\infty} \frac{\sin 2\pi F t}{\pi t} e^{-i\omega t} \, dt \right|^2, \qquad \omega = 2\pi f; \\ &= \begin{cases} N_0, & |f| < F; \\ 0, & |f| > F. \end{cases} \end{aligned} \tag{110}$$

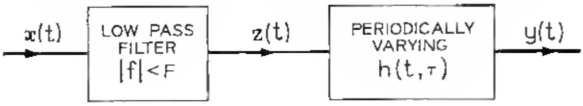

Fig. 4—Conversion of shot noise $x(t)$ to white noise $z(t)$.

Now hold F fixed and let $\nu \to \infty$. The individual pulses comprising $z(t)$ become smaller and smaller, and overlap more and more. In the limit $z(t)$ becomes zero-mean gaussian noise with the power spectrum (110).

Finally, let $F \to \infty$. Then $z(t)$ becomes white gaussian noise with the flat power spectrum $W_z(f) = N_0$. According to equation (107), the response of the Fig. 4 system at time t to a unit impulse applied at time τ tends to $h(t, \tau)$ as $F \to \infty$. Therefore, the results obtained in Sections VI, VII, and VIII for shot noise input in Fig. 2 are carried into corresponding results for white noise input (i.e., $x(t)$ in Fig. 2 is white gaussian noise) by the substitutions (108), namely $\nu E(a) = 0$ and $\nu E(a^2) = N_0$.

Setting $\nu E(a) = 0$ in equation (98) for $y_{\text{per}}(t)$ shows that $y_{\text{per}}(t)$ is zero for zero-mean white noise input. Consequently, $y(t)$ contains no dc or sinusoidal components.

Setting $\nu E(a) = 0$, and $\nu E(a^2) = N_0$ in equation (91) for $W_y(f)$ shows that the power spectrum of $y(t)$ is given by equation (105) when the input is white gaussian noise. Furthermore, $y(t)$ is composed entirely of $y_N(t)$; and equations (102), (103), and (104) become

$$\langle y^2(t) \rangle = N_0 \sum_{n=-\infty}^{\infty} \int_0^T h^2(t - nT, \tau)\, d\tau, \tag{111}$$

$$\begin{aligned} \overline{y^2(t)} &= \frac{N_0}{T} \int_0^T d\tau \int_{-\infty}^{\infty} dt\, h^2(t, \tau), \\ &= \frac{N_0}{T} \int_0^T d\tau \int_{-\infty}^{\infty} df\, |s(f, \tau)|^2. \end{aligned} \tag{112}$$

The fraction of time any particular member of the ensemble of outputs spends in the infinitesimal interval $Y < y(t) < Y + dY$ is

$$\frac{dY}{T} \int_0^T dt [2\pi \langle y^2(t) \rangle]^{-\frac{1}{2}} \exp\,[-Y^2/(2\langle y^2(t) \rangle)] \tag{113}$$

where $\langle y^2(t) \rangle$ is the function of t defined by equation (111).

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APPENDIX A

The Power Spectrum for Figure 1a

Here we give some of the steps leading from the first line to the second line of equation (42) for $W_{V_N}(f)$. The first line is

$$W_{V_N}(f) = \frac{\nu q^2}{T} \int_0^{\alpha T} |s(f,\tau)|^2 \, d\tau \tag{114}$$

where $s(f, \tau)$ is given by equation (31). Multiplying (31) by its complex conjugate gives

$$|s(f,\tau)|^2 = \frac{C^{-2}}{\gamma^2 + \omega^2} + \frac{C^{-2} b^2 e^{2\alpha\tau} \gamma^2}{|(\gamma + i\omega)\omega|^2} \left| \frac{z - z^\alpha}{1 - bz} \right|^2 + 2 \text{ Real} \left[\frac{C^{-2} b e^{\gamma\tau + i\omega\tau} (z - z^\alpha)(-\gamma)}{(\gamma - i\omega)(1 - bz)(\gamma + i\omega)(i\omega)} \right]. \tag{115}$$

Then

$$\int_0^{\alpha T} |s(f,\tau)|^2 \, d\tau = \frac{C^{-2}}{\gamma^2 + \omega^2} \left[\alpha T + \frac{b^2 (e^{2\gamma\alpha T} - 1)\gamma}{2\omega^2} \left| \frac{z - z^\alpha}{1 - bz} \right|^2 + 2 \text{ Real} \frac{b(e^{\gamma\alpha T + i\omega\alpha T} - 1)(z - z^\alpha)(-\gamma)}{(1 - bz)(\gamma + i\omega)(i\omega)} \right]. \tag{116}$$

Upon introducing the values $b = \exp(-\gamma\alpha T)$, $z = \exp(-i\omega T)$, and using the identity

$$\tfrac{1}{2}(1 - b^2) \left| \frac{z - z^\alpha}{1 - bz} \right|^2 = -\text{Real} \frac{(z^{-\alpha} - b)(z - z^\alpha)}{1 - bz} \tag{117}$$

the quantity within the square brackets in equation (116) becomes

$$\alpha T + \text{Real} \frac{(1 - bz^\alpha)(1 - z^{1-\alpha})\gamma(\gamma - i\omega)}{(1 - bz)\omega^2(\gamma + i\omega)} \tag{118}$$

and thus leads to the expression of $W_{V_N}(f)$ given by the second line of equation (42).

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